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Fifty-eight presents were announced as having been received since the last meeting.

The Motions of the Stars in a Kapteyn-Universe.

By J. H. Jeans.

1. In his recent paper "On the Distribution of the Stars in Space,"[*] and again in his "First Attempt at a Theory of the Arrangement and Motion of the Sidereal System,"[†] Professor Kapteyn has given a masterly summary of our present knowledge, mainly due to his own researches, of the arrangement of the stars in space. In brief, he finds that the arrangement of the stars may be regarded, with very tolerable accuracy, as consisting of a system of spheroidal shells, each of constant density. The spheroids are similar and similarly situated; they are oblate, having their axes in the ratio 5·102 : 5·102 : 1, their equal longer axes being in the galactic plane and their common centre being not far removed from the position of the sun.

In the second of these papers Professor Kapteyn attempts a dynamical investigation of such a system of stars. He assumes that the stars are in a dynamically steady state, and makes the further supposition that along the shortest axes of these spheroids (*i.e.* in the directions of the galactic poles) the law of distribution is Maxwell's well-known law for the distribution of molecular velocities of a gas in isothermal equilibrium.

Kapteyn's first assumption can neither be proved nor disproved; in the present state of our knowledge it is probably the most reasonable assumption to take as the basis of a dynamical discussion. But, for

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[*] *Astrophys. Journ.*, 52 (1920), 23.

[†] *Ibid.* (1922).

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reasons which will appear later, the second assumption seems much less plausible. Indeed, in introducing it Kapteyn writes: "We will now introduce some considerations from the kinetic theory of gases, the applicability of which to the stellar system might be considered doubtful. . . . The results which will be derived cannot lay claim to be demonstrably correct."

The main object of the present paper is to show that the introduction of Kapteyn's dubious second assumption is unnecessary. As we shall now see, the principal features of the motion of the stars can be deduced from a knowledge of their distribution in space, combined only with the single assumption that the system of stars is in a steady state.

2. Kapteyn's given distribution is one of symmetry about a plane (the Milky Way) and about an axis (the pole of the Milky Way). Taking the centre of the system as origin and the plane of the Milky Way as the plane $z=0$, we introduce cylindrical co-ordinates ϖ, θ, z, and denote the velocities of a star in the directions of ϖ, θ, z increasing by Π, Θ, Z.

Let the stars be of various distinct types, distinguished by the suffixes 1, 2, 3. . . .

Let the law of distribution of stars of type 1 at any instant t be assumed to be

$$f_1(\Pi, \Theta, Z, \varpi, \theta, z, t)d\Pi\, d\Theta\, dZ\, d\varpi\, \varpi\, d\theta\, dz \quad . \quad . \quad (1)$$

this representing the number of stars in the element of volume $d\varpi\, \varpi\, d\theta\, dz$ of which the velocity components at time t lie within the small range between Π and $\Pi + d\Pi$, Θ and $\Theta + d\Theta$, Z and $Z + dZ$.

Each star will be in motion under gravitational forces which may be regarded as made up of two parts—

(A) The forces from adjacent stars.

(B) The forces from the gravitational field of the universe as a whole.

A simple calculation shows that in our own system of stars forces (A) are in general quite insignificant in comparison with forces (B).* Thus we may expect a system of stars such as our own to tend first to assume a state of steady motion under forces (B). When such a state has been reached the average effect of forces (B) becomes zero, and forces (A) now come into play, until gradually the system assumes a final steady state under both forces (A) and (B). This final steady state is exactly analogous to Maxwell's steady state in the kinetic theory of gases; under it the velocities of the stars in any small region of space conform to Maxwell's law of distribution of velocities, so that these can show no preference for special directions in space, and there can be no star-streaming. The fact that star-streaming is observed in our universe affords conclusive proof that this final state has not yet been attained, a conclusion which is fully in accordance with calculations as to the time which would be required for its attainment.†

On the other hand, the time required for the attainment of a steady state under forces (B) alone is of an altogether lower order of magnitude, so that it is quite possible that a steady state of this kind has already

* *M.N.*, 74 (1913), 109.

† *L.c. ante*, p. 112.

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been attained. In agreement with Kapteyn, we shall assume this to be the case throughout the remainder of the present paper.

3. For such a steady state it is readily shown * that the law of distribution (1) assumes the special form

$$f_1[\tfrac{1}{2}(\Pi^2+\Theta^2+Z^2)-V,\ \varpi\Theta]d\Pi\, d\Theta\, dZ\, d\varpi\, \varpi\, d\theta\, dz \quad . \quad (2)$$

where V denotes the gravitational potential of the universe as a whole.

In this formula, f_1 is no longer specified as depending explicitly upon t, but it will be understood that the action of the feeble forces (A) results in a slow secular change in the constants which occur in the specification of f_1. It is easily shown that this formula demands star-streaming in the direction of Θ, the direction which Charlier and Kapteyn assign as the axis of star-streaming in our system.

Let ν_1 denote the number per unit volume of stars of type 1 in any small region of space, and let $(\Pi\Theta)_1$ denote the average value of the product $\Pi\Theta$ for these stars. Then, under the law of distribution (2),

$$\nu_1(\Pi\Theta)_1=\iiint_{-\infty}^{+\infty} f_1\Pi\Theta\, d\Pi\, d\Theta\, dZ=0,$$

and similarly

$$\nu_1(\Theta Z)_1=0, \quad \nu_1(Z\Pi)_1=0, \quad \nu_1(\Pi^2)_1=\nu_1(Z^2)_1.$$

Let us introduce quantities p_1, q_1 defined by

$$\nu_1(\Pi^2)_1=\nu(Z^2)_1=p_1, \quad \nu_1(\Theta^2)_1=q_1 \quad . \quad . \quad . \quad (3)$$

Taking u, v, w for the moment to represent components of stellar velocity along any three rectangular axes x, y, z, it is easy to deduce from the law of distribution (2) three equations of the type †

$$\frac{\partial}{\partial x}[\nu_1(u^2)_1]+\frac{\partial}{\partial y}[\nu_1(uv)_1]+\frac{\partial}{\partial z}[\nu_1(uw)_1]= {}_1\frac{\partial V}{\partial x} \quad . \quad . \quad (4)$$

these being, in point of fact, the hydrodynamical equations of motion for the ν_1 stars of the first type in a unit volume at x, y, z, taking into account the fact that forces (A) are negligible.

Transforming to the cylindrical co-ordinates ϖ, θ, z already employed, these equations become

$$\frac{\partial p_1}{\partial z}=\nu_1\frac{\partial V}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{\partial q_1}{\partial \theta}=\nu_1\frac{\partial V}{\partial \theta} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{\partial}{\partial \varpi}(\varpi p_1)-q_1=\varpi\nu_1\frac{\partial V}{\partial \varpi} \quad . \quad . \quad . \quad . \quad (7)$$

Since the field of stars is assumed to be symmetrical about the axis of z, $\partial V/\partial\theta$ must vanish, and equation (6) merely expresses the obvious

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* *Problems of Cosmogony and Stellar Dynamics*, p. 236.

† *Ibid.*, p. 232.

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fact that q_1 cannot vary with θ. It is convenient to rewrite equation (7) in the form

$$\frac{\partial p_1}{\partial \varpi}+\frac{p_1-q_1}{\varpi}=\nu_1\frac{\partial V}{\partial \varpi} \quad . \quad . \quad . \quad . \quad (8)$$

and equations (5) and (8) now contain the material for discussion.

The equations, as at present written, refer only to stars of one special type, but we are free to add together corresponding sides of any number of such equations. In this way we obtain equations such as

$$\frac{\partial p}{\partial z}=\nu\frac{\partial V}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{\partial p}{\partial \varpi}+\frac{p-q}{\varpi}=\nu\frac{\partial V}{\partial \varpi} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where p, q, ν now refer to all stars of any type or class desired, *e.g.* all B-type stars, or all luminous stars or all dark stars. But V is in every case the gravitational potential of all stars and of all dark matter in the universe. It will be noticed that equations (9) and (10) are no longer hydrodynamical equations.

Kapteyn's investigation gives us in effect the values of ν at every point of the universe. If we assume the stars to have everywhere the same known average mass, it is a mere matter of calculation to determine V at every point. This being done, equation (9) enables us to find p at each point, and equation (10) then determines q at each point. Since ν is already known, a knowledge of p and q gives the mean values of Π^2, Z^2, and Θ^2 at each point of space and for any special class of stars we please. The mean values of Π^2 and Z^2 will give us information as to the mean peculiar motions of the stars, while the mean value of Θ^2-Z^2 will give information as to the extent of star-streaming.

4. As a preliminary illustration of the use of our formulæ let us consider their application to the B stars. Charlier* has shown that these are confined to a flat, biscuit-shaped space surrounding the galactic plane. The sun is only about 90 parsecs from the centre of this region, its radius in the plane of the galaxy being about 500 parsecs and its extension on either side of this plane only about 150 parsecs. As a matter of convenience let us suppose the boundary of this region to coincide with the first ellipsoidal shell of Kapteyn, of which the boundary is

$$\frac{\varpi^2}{a^2}+\frac{z^2}{c^2}=1, \quad \text{where} \begin{cases} a=602 \text{ parsecs} \\ c=118 \text{ parsecs} \end{cases} \quad . \quad . \quad (11)$$

Let us further, as an approximation which appears to be in very fair agreement with the facts, assume the B-type stars to be arranged in similar ellipsoidal shells inside this one, so that the density ν of B-type stars is of the form

$$\nu=\phi\left(1-\frac{\varpi^2}{a^2}-\frac{z^2}{c^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

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[*] *Lund Meddel.*, 2 (1916), No. 14.

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Inside the ellipsoid occupied by the B-type stars the potential of the whole universe, as arranged by Kapteyn, is of the form

$$V = C - A\varpi^2 - Bz^2,$$

where the last two terms arise from the potential of the ellipsoid itself and all external ellipsoids add only a constant term to V.

From equation (9) we now have

$$p = \int \nu \frac{\partial V}{\partial z} dz = -\int \phi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right) 2Bz\, dz \quad . \quad . \quad (13)$$

If we put $\int \phi(x)\, dx = \psi(x)$ and adjust the additive constant so that $\psi(x)$ vanishes when $x = 0$, then equation (13) becomes

$$p = Bc^2 \psi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right) \quad . \quad . \quad . \quad . \quad (14)$$

no function of ϖ being added, since p must vanish at the boundary (11). Equation (10) now gives

$$\frac{q-p}{\varpi} = \frac{2\varpi}{a^2}(Aa^2 - Bc^2)\phi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right)$$

or

$$\frac{q-p}{\nu} = 2(Aa^2 - Bb^2)\frac{\varpi^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The maximum value of p occurs at the centre of the system, being $Bc^2\psi(1)$ or $Bc^2 \int_0^1 \phi(x)\, dx$. If we assume, for illustration, a law of density

$$\nu = \nu_0\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right),$$

then $\phi(x) = \nu_0(x)$ and $\psi(1) = \frac{1}{2}\nu_0$. In this case the value of p/ν at the centre is $\frac{1}{2}Bc^2$. Had we assumed uniform density we should have had $p/\nu = Bc^2$ at the centre.

In his numerical calculations Kapteyn takes the parsec as unit of length and the gravitational mass of the "average star" as unit of mass. In these units he calculates the gravitational forces at the points $(a, 0)$ and $(0, c)$, and finds

$$2Aa = 33{\cdot}19, \qquad 2Bc = 40{\cdot}06.$$

In these units the value of p/ν at the origin is

$$p/\nu = \tfrac{1}{2}Bc^2 = 1182.$$

To reduce to ordinary C.G.S. units we have to multiply by $\gamma M/l$. Here l is the number of cms. in a parsec ($3{\cdot}08 \times 10^{18}$), γ is the gravitation constant ($6{\cdot}66 \times 10^{-8}$), and M is the mass of the "average star" in grammes. More exactly this last quantity should be the mass of the average visible star increased by the ratio of dark matter to visible stars

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in the universe. Later on we shall determine the value of M as $6{\cdot}34 \times 10^{33}$ grammes. We shall use this value here, so that

$$\frac{\gamma \mathrm{M}}{l} = 1{\cdot}37 \times 10^8,$$

and

$$\frac{p}{\nu} = \tfrac{1}{2}\mathrm{B}c^2 = 16{\cdot}2 \times 10^{10} \text{ (cms. per sec.)}^2.$$

This quantity is of course equal to the mean value of Z^2 or Π^2 for all the B-type stars near the origin. We see that even close to the origin, where they attain their maximum values, Z and Π are only of the order of 4 kms. a second (or 6 kms. a second in the extreme case of uniform distribution). It is at once clear that the low velocities of the B-type stars are fully accounted for by their observed distribution in space: it is not necessary to interpret them as a consequence of their supposed exceptional mass.

Independently of the law of distribution of density, equation (15) gives

$$\frac{q-p}{\nu} = 15{,}252\left(\frac{\varpi}{a}\right)^2 = 209 \times 10^{10}\left(\frac{\varpi}{a}\right)^2 \text{ (cms. per sec.)}^2.$$

This quantity represents the mean value of $\Theta^2 - Z^2$ or $\Theta^2 - \Pi^2$. When it has a positive value it is clear that there must be star-streaming in the directions of θ increasing and decreasing. If U′, U″ are the velocities of the two star streams, we shall see later (§ 6) that $\frac{q-p}{\nu}$ is equal to U′ U″. Thus in the special case now under discussion the velocities of the star streams are of the order of 14·5 kms. a second at the boundary $\varpi = a$ (602 parsecs) and decrease uniformly with ϖ, precisely as though the two star streams were rigid bodies rotating in opposite directions. At Charlier's position for the sun ($\varpi = 90$ parsecs), the velocity of star-streaming for the B-type stars will be only about 2·2 kms. a second.

5. Let us now apply the same method to the general mass of the stars. From equation (9) we have

$$p = -\int \nu \frac{\partial \mathrm{V}}{\partial z}\, dz \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where the integration is along a line parallel to the axis of z until a region is reached at which the star density ν is negligible. The integral can only be evaluated by quadratures. Kapteyn's investigation gives the value of ν at every point, and he has calculated the value of $-\partial \mathrm{V}/\partial z$, which he denotes by $\mathrm{G}\cos\phi$, at certain selected points. I have first evaluated p at points along the galactic axis ($\varpi = 0$), taking the thicknesses of Kapteyn's various "shells" as the elements of integration in equation (16). The calculation is exhibited in Table I. The entries Ex. 1, Ex. 2, etc., refer to shells obtained by extrapolation beyond the ten shells of Kapteyn. No estimate is made of the density or thickness of these shells, the extrapolation being only with reference to their contribution to p. The last column gives the square root of the mean value

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of Z^2 or Π^2 in kilometres per second on the assumption that M has the value already assigned to it in § 4. This may be regarded as giving approximately one component, other than Θ, of the star's peculiar velocity; it is seen to range from about 13 kms. a second in the close neighbourhood of the origin, through a value of 17 kms. a second at about 300 parsecs from the origin, to a final value of about 23 kms. a second, which is approximately attained at about 1600 parsecs from the origin.

Table I.

Values of p and Z along Galactic Pole.

Shell.	Mean ν.	Mean $\frac{\partial V}{\partial z}$.	dz in parsecs.	$\nu\frac{\partial V}{\partial z}dz$.	Ellipsoid.	z in parsecs.	p.	ν.	$\frac{p}{\nu}=(Z^2)$.	Z (kms. per se
1	·036	22·0	118	93·5	Centre	0	504·0	·0451	11,200	12·4
2	·023	48·0	80	88·3	I.	118	410·5	·0285	14,400	14·0
3	·014	58·5	98	80·3	II.	198	322·2	·0180	17,900	15·7
4	·0090	64·8	117	68·2	III.	296	241·9	·0113	21,400	17·1
5	·0057	66·3	140	52·9	IV.	413	173·7	·00715	24,300	18·2
6	·0036	65·0	164	38·4	V.	553	120·8	·00451	26,800	19·2
7	·0023	62·3	185	26·5	VI.	717	82·4	·00285	29,000	19·9
8	·0014	58·5	212	17·4	VII.	902	55·9	·00180	31,100	20·7
9	·0009	54·0	251	12·2	VIII.	1114	38·5	·00113	34,100	21·6
10	·0006	48·7	295	8·6	IX.	1365	26·3	·000715	36,800	22·4
Ex. 1				6·1	X.	1660	17·7	·000451	39,300	23·2
Ex. 2				4·2	Ex. I.		11·6	·000285	40,700	23·6
Ex. 3				2·9	Ex. II.		7·4	·000180	41,100	23·7
					Ex. III.		4·5	·000113	40,000	23·4

The next stage in the computations was to construct tables giving the values of p throughout the whole system. The method adopted was to regard p at any point, as before, as arising from contributions from all the shells standing outside this point. It was found that the contribution $\nu\frac{\partial V}{dz}dz$ from any specified shell did not vary greatly for different points of this shell until very low galactic latitudes were reached. For instance, for shell 2, if the contribution in latitude 90 is taken to be 1·000, those in latitudes 75°, 60°, 45°, 30°, 15°, and 0° are respectively ·999, ·996, ·990, ·97, ·90, and ·67; for shell 8 the corresponding numbers are ·984, ·950, ·86, ·745, ·57, and ·27. From a table giving the ratio of contribution in any latitude to that in latitude 90° it was found easy to evaluate p at any desired point. The following table gives the values found for p and p/ν at a series of points in the galactic plane, the points selected being the boundaries of the various shells. The table also contains the value of Z, the square root of (Z^2) in kilometres per second, the value of $(q-p)/\nu$ calculated from equation (10), and the square root of $(\Theta^2 - Z^2)$ in kilometres per second.

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TABLE II.

Values of p, Z, q, and Θ in Galactic Plane.

Ellipsoid.	ϖ in parsecs.	ν.	p.	$\frac{p}{\nu}$.	$\frac{q-p}{\nu}$.	Z (kms. per sec.).	$\sqrt{(\Theta^2-Z^2)}$ (kms. per sec.).
Centre	0	·0451	504	11,200	0	12·4	0
I.	602	·0285	368	12,900	14,300	13·3	14·0
II.	1010	·0180	251	13,900	25,900	13·8	18·8
III.	1510	·0113	165	14,600	38,500	14·1	23·0
IV.	2106	·00715	102	14,400	51,500	14·0	26·6
V.	2820	·0045	63	14,000	62,300	13·8	29·2
VI.	3656	·00285	36	12,600	69,100	13·1	30·8
VII.	4600	·00180	21	11,600	78,200	12·6	32·7
VIII.	5675	·00113	12	10,600	83,400	12·1	33·8
IX.	6960	·000715	7	9,900	86,300	11·7	34·4
X.	8465	·00045	4	8,900	86,300	11·0	34·4

The tables have given the values of p, Z, q, Θ only in the galactic plane and along the galactic axis. It is easy to obtain a rough idea of the values of these quantities in other parts of space. As regards p, we have already noticed that the contribution of any individual shell to p remains approximately constant until low galactic latitudes are reached, so that we may think of p as remaining constant over any shell except in low galactic latitudes. In other words, except in these low latitudes p and Z may be regarded as functions of ν only, and the last four columns of Table I. may be regarded as tabulating p and Z in terms of ν. In low galactic latitudes Table II. may, of course, be used in a similar manner. As regards q, we obtain from equations (9) and (10) on elimination of V

$$\frac{\partial}{\partial z}\left(\frac{q-p}{\nu\varpi}\right)=\frac{\partial p}{\partial \varpi}\frac{\partial}{\partial z}\left(\frac{1}{\nu}\right)-\frac{\partial p}{\partial z}\frac{\partial}{\partial \varpi}\left(\frac{1}{\nu}\right).$$

We have seen that in fairly high galactic latitudes p may be regarded as a function of ν, in which case the right hand of this equation vanishes. Again, in low latitudes $\partial p/dz$ and $\partial \nu/dz$ are very small, so that again the right-hand member is small. We may get a rough idea of the extent of star-streaming by neglecting the right-hand member of this equation altogether, in which case we find that $(q-p)/\nu$ becomes a function of ϖ only, and Table II. may be regarded as giving this quantity in terms of ϖ.

6. If we regard the stars in any small region of space as forming two star drifts in the direction of θ increasing and decreasing, if ν', ν'' are the densities of these streams, U′, $-$U″ the stream velocities, and if u', v', w' and u'', v'', w'' are the peculiar velocities of individual stars in these streams, then

$$p=\sum v'^2+\sum v''^2 \text{ in unit volume}$$
$$=\nu\overline{v^2}$$
$$q=\sum(\mathrm{U}'+u')^2+\sum(-\mathrm{U}''+u'')^2=\nu'\mathrm{U}'^2+\nu''\mathrm{U}''^2+\nu\overline{u^2}.$$

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In these equations we have $\nu' + \nu'' = \nu$, $\nu'U' = \nu''U''$ and $\overline{u^2} = \overline{v^2}$, so that

$$\frac{q-p}{\nu} = \frac{\nu'U'^2 + \nu''U''^2}{\nu} = U'U''.$$

If h_1, h_2 are the semiaxes of the Schwarzschild velocity ellipsoid at any point, we have

$$\frac{h_2^2}{h_1^2} = \frac{\Sigma\Theta^2}{\Sigma Z^2} = \frac{q}{p}.$$

7. In the neighbourhood of our sun we may take approximately $h_1/h_2 = 0{\cdot}577$, giving $h_2^2/h_1^2 = 3{\cdot}00$, or

$$\frac{q-p}{\nu} = \frac{p}{\nu} \times 2{\cdot}00.$$

In Table II. this relation is seen to be satisfied at $\varpi = 1090$ parsecs, at which $p/\nu = 14{,}000$ and $(q-p)/\nu = 28{,}000$. On the two-drift hypothesis, the numbers of stars in the two drifts are found to be almost exactly in the ratio 3 : 2, and the relative velocity of the two drifts is 40 kms. a second. Thus $U' = 16$ kms. a second and $U'' = 24$ kms. a second, giving

$$U'U'' = 384 \times 10^{10} \text{ C.G.S. units.}$$

We may notice in passing that the value of $U'U''$ is not very sensitive to the ratio in which the stars are supposed divided between the two streams. If we had supposed the streams to be equal, we should have had $U' = U'' = 20$ kms. a second, and the value of $U'U''$ would have altered only to 400×10^{10}.

We have now found that the value of $(q-p)/\nu$ in Kapteyn units is 28,000 and in C.G.S. units is 384×10^{10}. Hence, as in § 4, we must have

$$384 \times 10^{10} = \frac{\gamma M}{l} \times 28{,}000,$$

whence, on putting $\gamma = 6{\cdot}66 \times 10^{-8}$, $l = 3{\cdot}08 \times 10^{18}$, we find

$$M = 6{\cdot}34 \times 10^{33} \text{ grammes,}$$

which is the value which has been used throughout the paper, and has already been inserted in the values of Θ and Z given in Tables I. and II.

This value, being 3·2 times the mass of the sun, is obviously too large to represent the mass of a single star. The average mass of binary systems is found to be about 1·6 times the sun's mass, but these are mainly binaries of long period whose origin must probably be traced to separate nuclei in the original nebulæ, and whose combined mass may therefore be expected to be twice that of the average single star. Assuming the average star to have a mass of 0·8 times the sun's mass, our value of M must be interpreted as showing that if the stars are in a steady state, there must be about three dark stars in the universe to every bright star.

8. Subject to our being able to assume the existence of this very appreciable amount of dark matter, it appears that the observed motions

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of the stars can be explained if, and only if, we are free to assume our sun to be at a distance of 1090 parsecs from the centre of the galaxy. This figure of 1090 parsecs cannot be regarded as having any great degree of accuracy, for we have made use of Kapteyn's observational distribution of density, which was derived only on the supposition that the sun is approximately at the centre of the system. The true value of the sun's distance from the centre can only be obtained as the result of a more refined investigation of stellar density, of which the need is fully recognised by Professor Kapteyn, in which full account will be taken from the outset of the excentric position of the sun.

9. Without carrying out this investigation in detail, it is easy to discover the general nature of the change to be expected in our estimate of 1090 parsecs. In place of the quantities z, ϖ, ν, p, q, M so far used, let us transform to new quantities $z', \varpi', \nu', p', q'$, M$'$ connected with the foregoing by the relations

$$z' = k_1 z, \quad \varpi' = k_1 \varpi, \quad \nu' = k_2 \nu, \quad p' = k_3 p, \quad q' = k_3 q, \quad \mathrm{M}' = k_4 \mathrm{M}$$

where k_1, k_2, k_3, k_4 are constants to which a value will shortly be assigned. Transformed to these quantities, our fundamental equations (9) and (10) become

$$\frac{\partial p'}{\partial z'} = \frac{k_1 k_3}{k_2^2 k_4} \nu' \frac{\partial \mathrm{V}'}{\partial z'}; \qquad \frac{\partial p'}{\partial \varpi'} + \frac{p' - q'}{\varpi'} = \frac{k_1 k_3}{k_2^2 k_4} \nu' \frac{\partial \mathrm{V}'}{\partial \varpi'}.$$

Thus the fundamental equations transform back to themselves if $k_1 k_3 = k_2^2 k_4$. If this relation is satisfied between the four k's there will be a solution in z', ϖ', etc., corresponding exactly to the solution we have already found in z, ϖ, etc. To what extent is it possible that this new solution should represent the observed facts after allowing for the excentric position of the sun?

The new solution gives a distribution of mass which is of course precisely similar to the old, except for proportional changes in length, mass, and density. To account for the observed ratio of axes of the Schwarzschild ellipsoid, our sun must be at the point in the new solution corresponding to $\varpi = 1090$ parsecs in the old solution; this will be given by $\varpi' = 1090$. To account for the observed density of stars near our sun we must have $\nu' = 0{\cdot}0451$ in the new solution as against $\nu = 0{\cdot}168$ in the old, so that $k_2 = 2{\cdot}68$. To account for the observed actual velocity of the stars in kilometres per second we must have $p'\mathrm{M}'/\nu' = p\mathrm{M}/\nu$, so that we must have $k_3 k_4 = k_2$. The new solution would give outer shells of the same shape as those of the old solution, but to make the observed density accord with observation a suitable value of k_1 must be chosen. Thus at $\varpi' = 6960$ the new solution gives $\nu = {\cdot}000715 \times 2{\cdot}68 = {\cdot}001910$, which observation shows to correspond to a distance of $\varpi = 4500$. Thus we must take $k_1 = 6960 \div 4500 = 1{\cdot}55$, and it will be found that this value of k_1 together with the previously found value of $k_2 = 2{\cdot}68$ gives a new solution which represents the observed distribution of stars in the outer layers with tolerable accuracy. Eliminating k_3 from the two relations $k_1 k_3 = k_2^2 k_4$ and $k_3 k_4 = k_2$ already found, we obtain $k_4^2 = k_1 / k_2$, so that $k_4 = 0{\cdot}76$.

Thus in the new solution the sun's position ($\varpi' = 1090$ parsecs)

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will be $\varpi = 700$ parsecs, and the average mass M will be $4 \cdot 8 \times 10^{33}$ grammes. Taking the average mass of a single star as 0·8 times the sun's mass, this indicates two dark stars to each bright star. In the new solution the stellar density at the sun's position ($\varpi = 700$ parsecs) will be 0·0451, the value assumed by Kapteyn, and the motion, as regards both star-streaming and peculiar motions of the stars, will be that actually observed in the neighbourhood of the sun. The new solution will represent Kapteyn's observational material well enough as regards the outer shells of stars, but the agreement with observation as regards the inner shells needs careful discussion. For reasons which will be found in his own paper, Kapteyn concludes that the greatest value of ϖ which can be reconciled with observation is in the neighbourhood of 700 parsecs. Thus our estimate of ϖ is just about at the limit permitted by observation. If the estimate had fallen entirely outside the limit, we should of course have had to conclude that the stars had not yet attained a steady state of the type assumed in this paper.

In any case we may notice that our estimate of $\varpi = 700$ parsecs places the centre of the galaxy right outside the system of the B stars as determined by Charlier. This leads us to regard the B stars rather as a moving star-cluster than as an integral part of the galactic system in steady motion with the rest of this system. We shall return to a consideration of this point of view in the next paper.

The Dynamics of Moving Clusters.

By J. H. Jeans.

1. In a recent "Research on Moving Clusters,"[*] N. H. Rasmuson has provided material for testing a suggestion I put forward some years ago[†] that a moving cluster ought to assume a flattened shape as a consequence of encounters with other showers of stars. In particular, if the stars met by the cluster are moving at random the cluster ought to experience a flattening in the direction of its own motion. Certain of the clusters studied by Rasmuson (the Ursa Major cluster and that in Perseus, to quote his own instances) show a flattening of the type predicted, but others (*e.g.* the Scorpio-Centaurus cluster) show exactly the opposite effect.

In my investigation the forces acting on a single star of a moving cluster were supposed divided into three parts :

(*a*) The forces acting from the remaining stars of the cluster ;
(*b*) The forces from stars which are undergoing an "encounter" with this cluster, in the sense of the forces between single stars being of appreciable amount ;
(*c*) The forces from all the remaining stars of the universe.

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[*] *Lund Meddel.*, 2, No. 26.

[†] *M.N.*, 76 (1916), 552.